

**Thermodynamics with long-range interactions: From Ising models to black holes**

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Methods are presented which enables one to analyze the thermodynamics of systems with long-range interactions. Generically, such systems have entropies which are nonextensive (do not scale with the size of the system). We show how to calculate the degree of nonextensivity for such a system. We find that a system interacting with a heat reservoir is in a probability distribution of canonical ensembles. The system still possesses a parameter akin to a global temperature, which is constant throughout the substance. There is also a useful quantity which acts like a *local temperatures* and it varies throughout the substance. These quantities are closely related to counterparts found in general relativity. A lattice model with long-range spin-spin coupling is studied. This is compared with systems such as those encountered in general relativity and gravitating systems with Newtonian-type interactions. A long-range lattice model is presented which can be seen as a black hole analog. One finds that the analog's temperature and entropy have many properties which are found in black holes. Finally, the entropy scaling behavior of a gravitating perfect fluid of constant density is calculated. For weak interactions, the entropy scales like the volume of the system. As the interactions become stronger, the entropy becomes higher near the surface of the system, and becomes more area scaling.

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**I. INTRODUCTION**

In the study of thermodynamics, it is almost always implicitly assumed that the system does not possess long-range interactions. Very little is known about the thermodynamics of systems which do possess long-range interactions, except in special cases such as plasmas where the electromagnetic interactions are screened or systems which have no overall charge [1]. In both these instances, one can use standard thermodynamics, since effectively there is no long-range interaction. If however, the long-range interactions are not screened, then difficulties are encountered, such as the non-existence of the canonical ensemble [2] or inequivalence of microcanonical and canonical ensembles, and potential lack of a stable equilibrium configuration [3,4]. The latter is sometimes attributed to negative heat capacities [5]. Negative heat capacities are not only present in astrophysical systems, but have even been observed in fragmenting nuclei [6] and atomic clusters [7]. It is not known how to deal with these systems generically, although there have been some attempts to understand them outside of standard thermodynamics using the Tsallis entropy [8] (cf. also Ref. [9]).

A principle motivation for this work is therefore to provide a framework in which to study such systems. A second motivation comes from the study of black-hole thermodynamics. There it is found that the black-hole possesses an entropy which has unusual properties. Here, we will show that these properties are not limited to the black hole, but that other systems with long-range interactions exhibit related behavior. We will essentially construct an analog of a black hole by adding long-range interactions to a spin-lattice model.

Systems with long-range interactions are often referred to as *nonextensive* because the entropy and energy do not scale with the volume of the system. Normally, if one has a thermodynamical system and one holds the intensive variables

(temperature, pressure, and chemical potential) fixed, then if the size of the system is doubled, the extensive variables (entropy and energy) will also double. This is not true if the interactions are long range.

The purpose of this paper is to develop methods and a formalism to explore a number of facets of such nonextensive systems. We shall employ a principle of "local extensivity" which enables one to define thermodynamical quantities for an interacting system. We shall also show how to classify the degree of nonextensivity of the system by calculating the scaling behavior of the entropy as a function of total energy. The motivation for this part of the study comes from general relativity. There it is found that the entropy of a black hole is proportional to its area, rather than its volume (i.e., the entropy is nonextensive). In this study, we will see that this is a generic property of interacting systems, rather than something unique to the black hole.

We will also see that generically, a system interacting with a reservoir is not at a particular temperature, but rather is in a probability distribution of temperatures. This will be found by studying a system interacting with a reservoir in the microcanonical ensemble, i.e., the total energy of the system plus reservoir is fixed. Usually, if one then only looks at the system, it will be in a canonical ensemble (fixed temperature). When interactions are present, this will not be the case. This leads us to introduce a new type of ensemble, which we call the microlocal ensemble. It is equivalent to the microcanonical ensemble when there are no interactions.

Despite the fact that a system is not found in the canonical ensemble, we shall see that one can define a quantity we call the *global temperature*  $\beta_o$ . It describes the system as a whole, and is written in terms of the total energy, including the interacting terms. There is also a *local temperatures*, a quantity inspired by general relativity. Both types of temperature are measurable in principle.

In the case of short-range interactions, two systems

brought into thermal contact will be at the same temperature. Here, we will show that also for interacting systems, the global temperature of two systems is the same. This justifies to some extent, our use of the term temperature to describe  $\beta_o$ . However, we will see that if one has two separated systems with the same global temperature, then when they are brought into contact adiabatically, they will reach a new global temperature.  $\beta_o$  is therefore not an intensive quantity. We will also see that the local temperatures of the two systems are, in general, different for each system when in thermal equilibrium. Such an effect is analogous to the Tolman relation [10] which exists in curved space. There one finds a temperature gradient due to the curvature of space-time. Essentially, frequencies are redshifted by curvature. Since the temperature gives the probability distribution of a frequency spectrum, it is also redshifted. Here, we see that such an effect does not exist solely in curved space-time, but can also be thought of as due to the presence of long-range interactions.

To make our discussion more concrete, we will examine a toy model consisting of a lattice of spins in a magnetic field and interacting via a spin-spin coupling. However, rather than only nearest neighbor interactions, we will also consider the long-range couplings. We will consider the case of a uniform long-range interaction, as well as the case of two different systems interacting via two unequal uniform interactions. Such a situation arises when one considers two lattice clusters which are of small size. We will also discuss the continuum situation, where the interaction can be arbitrary and varies from site to site.

We then consider thermodynamics in the general theory of relativity. Comparisons between the lattice model and of black hole thermodynamics provide another motivation for this study. Previously, analogs of black holes [11,12] (so-called, acoustic, or solid state black holes) have been used to understand black hole radiation. However, they are not useful for understanding the black hole entropy. Here, we see that one can construct a black hole analog that can be used to study black hole entropy. One finds that the entropy can be nonextensive, just like a real black hole. We find for the analog that there is an infinite redshifting between its local temperature and its global temperature which has exactly the same form as a black hole. At exactly the point where the systems acts like a black hole, a degeneracy in the local energy levels forms. This degeneracy is universal, in the sense that it only depends on the form of the interaction. The universality is somewhat reminiscent of the universality of black hole entropy.

We will also investigate other gravitational systems in general relativity. In particular, we look at the entropy scaling behavior of a gravitating perfect fluid. The motivation for this comes partly from an earlier study [13] where it is shown that the black hole is not the only gravitating system which has an area-scaling entropy. A system of shells has an entropy that scales as the volume when the gravitational interaction is weak, but the entropy becomes area scaling at the point before a black hole is formed. Here, in looking at the gravitating perfect fluid, we find related behavior. We can look at the entropy scaling behavior not just in limiting

cases, but for all strengths of the gravitational interaction. As the strength of the gravitational interaction is increased, the entropy slowly moves to the outer surface of the perfect fluid. One finds that the total entropy is nonextensive, just like in a black hole, and approaches area-scaling behavior as the strength of the gravitational interaction gets stronger.

We also explore gravitating systems in the context of Newtonian-type dynamics. This is done to show that the redshifting of temperatures—usually considered to be an effect related to the curvature of geometry—also exists in other gravitational models which are not geometric theories.

The paper is organized as follows. In Sec. II, we introduce our formalism. First, in Sec. II A, we introduce our assumptions, which we call *locality* and *local extensivity* and show that these assumptions are obeyed by a number of common systems. Next, in Sec. II B, we use the microcanonical ensemble to show that a system interacting with a reservoir will not be found at a particular temperature, but rather will be in a probability distribution of different temperatures. Nonetheless, there is a parameter which behaves very similarly to a temperature, which we call the global temperature. This is defined in Sec. II C. The physical significance of the global temperature, as well as another parameter called the local temperature is explored in Secs. II D and II E. Then, in Sec. II F, we show that the global temperatures of two systems brought into contact are equal at equilibrium. The local temperatures need not be equal (an effect reminiscent of redshifting which is usually considered to be the sole domain of general relativity). This allows us to study lattice models where the long-range interaction is not uniform. Next, in Sec. III, we explore in some detail a lattice model with long-range interactions. In Sec. IV, we show that such a system can be made into an analog of a black hole and has a temperature and entropy with many properties reminiscent of black holes. In Sec. V, we show how one can generically calculate the entropy scaling behavior of an interacting system. This is done for a gravitating perfect fluid in Sec. VI. We find that the entropy becomes more area scaling as the gravitational interaction gets stronger. We conclude with some general remarks in Sec. VII and point to some open questions.

In Appendix B, we look at systems under the influence of Newtonian-type gravity and show that an analog of the Tolman relation exists—local temperatures are red shifted. In Appendix C, we look at more general interactions and go to the continuum limit.

## II. A FORMALISM FOR SYSTEMS WITH LONG-RANGE INTERACTIONS

### A. Locality and local extensivity

Let us consider two interacting systems 1 and 2 with total energy

$$m = E_1 + E_2 + G(E_1, E_2). \quad (1)$$

Here,  $G(E_1, E_2)$  is some interaction potential (which may include self-interacting terms) and  $m$  is the total energy [14]. In the absence of the interaction  $G$ , the energy of each sys-

tem would be  $E_1$  and  $E_2$ . We will refer to  $E_1$  and  $E_2$  as the *local energy*. Likewise,  $E = E_1 + E_2$  is the local energy of both systems. In other words,  $E$  can be thought of as the extensive part of the energy (or noninteracting part).

In general, a complicated system will include many such systems interacting together, or will be a continuum of separate systems at each point in space. For the purposes of illustration, we will consider the simple case where we can divide the total system into two parts. We will later consider more complicated setups.

We now make two assumptions.

*Locality.* The state of each system is determined only by its local energy  $E_i$  and local variables  $q_i$ .

*Local extensivity.* Correlations due to the interaction can be completely described by correlations in the energies  $E_i$ .

The first assumption is rather generic. The latter assumption is true for two systems interacting via a potential which can be put into the form of Eq. (1). The essential requirement for local extensivity is that if the potential introduces correlations in microscopic variables  $q_i$ , then it will also result in correlations between the local energies  $E_i$ . Correlations which depend on other thermodynamical quantities can also be described using this formalism. The logic behind these assumptions should become clear in a moment when we consider an example, but an important consequence is that if we write the total entropy of the two systems as  $S(E_1, E_2)$  then

$$S(E_1, E_2) = S_1(E_1) + S_2(E_2), \quad (2)$$

where  $S_i$  is the entropy of each subsystem. This relation is somewhat counterintuitive because when interactions are present one expects there to be correlations between the two systems, and therefore, one does not expect the entropy to be additive. However, the entropy is only additive because we have written it in terms of the local energies. The entropy does not scale linearly with the total energy  $m$  and is therefore nonextensive. Essentially, for fixed  $m$  there are correlations which exist because  $E_1$  is not independent of  $E_2$ , but once you specify  $E_1$  and  $E_2$  you have completely specified each subsystem.

Equation (2) follows from our assumptions because if systems 1 and 2 are only determined by local variables, then specifying local variables, such as  $E_1$  and  $E_2$  determines the number of possible states of each system. Furthermore, since the correlations between the two systems are only correlations between values of  $E_1$  and  $E_2$ , then once  $E_1$  and  $E_2$  are specified there are no additional correlations which would destroy the additivity of the entropy as given by Eq. (2).

To make this point clear, let us illustrate it with an example; consider a lattice of  $N$  spins with total energy

$$m = \sum_j h_j \sigma_j - \sum_{\langle jk \rangle} J_{jk} \sigma_j \sigma_k, \quad (3)$$

where the  $\sigma_j$  represent the spin at each lattice site (with values  $\pm 1$ ) and the  $h_j$  are magnetic field values (or internal energy levels). The  $J_{jk}$  are spin-spin coupling constants. In the standard Ising model, one takes the sum such that  $\langle jk \rangle$  are pairs of nearest neighbors. Here, we consider the case

where the spin-spin coupling is strong enough (or the lattice spacing is small enough), such that  $J_{jk}$  and  $h_j$  are relatively constant over a large region. For simplicity, we will imagine that the system is composed of two such regions separated by a short distance. This then gives (up to a constant)

$$m = h_1 e_1 + h_2 e_2 - J_1 e_1^2 / 2 - J_2 e_2^2 / 2 - J_{12} e_1 e_2, \quad (4)$$

where the dynamic variable  $e_i$  is the number of up spins minus the number of down spins inside each region, and  $J_i$  and  $h_i$  are the coupling inside each region and are known constants.  $J_{12}$  is the coupling between each region, and would presumably be smaller than the  $J_i$ . The number of sites in each region is assumed constant.

Now it is clear that our assumptions and Eq. (2) hold. Since the local energy of each system is  $E_i = h_i e_i$ , specifying  $E_i$ , completely fixes the number of up and down spins in each region. Furthermore, once  $E_1$  is specified, then the state of system 1 is completely independent of the second system, i.e., once  $E_1$  is specified, the state of system 1 has been determined (macroscopically). This state will now not depend on what value of  $E_2$  system 2 happens to have. Different values of  $E_2$  will, of course, mean that the effective magnetic field that system 1 feels will be different, but we have already specified  $E_1$ , so its macroscopic state will not change. Its microscopic state would not change either, since the interaction of Eq. (4) does not introduce any distinction between different microscopic configurations. Specifying  $E_i$  is the same as specifying  $e_i$  since the  $h_i$  are known. For each system, once the spin excess  $e_i$  is specified, then all spin combinations consistent with this value of  $e_i$  are equally likely. Once again, once  $e_1$  is specified for system 1, nothing depends on what happens with system 2 insofar as which states will be occupied.

The entropy of each system  $S_i$  is just given by the number of independent ways of arranging the spins (since each arrangement is equiprobable), i.e.,

$$S_i(E_i) = - \frac{N_i + e_i}{2} \ln \frac{N_i + e_i}{2N_i} - \frac{N_i - e_i}{2} \ln \frac{N_i - e_i}{2N_i}. \quad (5)$$

There are, of course, still correlations between the two systems—for fixed  $m$ , the number of up spins in system 2 will completely determine the number of up spins in system 1.

We will see that these assumptions also hold in the context of general relativity, but for the moment, let us return to the generic case and define the temperature. One has to be careful because as we will see the canonical ensemble does not exist.

## B. Multiple temperatures and the microlocal ensemble

The usual derivation of the canonical ensemble follows from considering a large reservoir  $\mathcal{R}$  in contact with a smaller system  $\mathcal{S}$ . One fixes the total energy of the combined system (hence, one is operating in the microcanonical ensemble), but we let energy flow between  $\mathcal{R}$  and  $\mathcal{S}$ . One then finds that the probability distribution of energies of  $\mathcal{S}$  is independent of the details of the reservoir. The distribution

depends on a quantity which is defined as the temperature and this defines the canonical ensemble. Here, we essentially repeat the standard derivation, except we have the interaction term given in Eq. (1). We will find that the distribution looks very different. One can think of the system as being in a probability distribution of different canonical ensembles.

Although we could work in the microcanonical distribution from the very start, it will prove useful to define a new ensemble which we will call the *microlocal ensemble*. Rather than fixing the total energy  $m$ , we shall fix the total local energy  $E = E_{\mathcal{R}} + E_{\mathcal{S}}$ . The motivation for this should become clear as we proceed. In the case when the interaction  $G$  is zero, the microlocal ensemble and the microcanonical ensemble are clearly identical.

Let us therefore consider two system with fixed local energy  $E$ , and imagine that  $\mathcal{R}$  is very large, and constitutes a reservoir, i.e.,  $E_{\mathcal{S}} \ll E_{\mathcal{R}}$ . We then allow energy to flow between  $\mathcal{S}$  and  $\mathcal{R}$  until the systems reach equilibrium. At equilibrium, and for large systems, one is most likely to find the system in a state which maximizes the entropy.

The probability that  $\mathcal{S}$  has energy  $E_{\mathcal{S}}$  for a fixed  $E$  is given by counting the number of possible states of the system when  $\mathcal{S}$  has local energy  $E_{\mathcal{S}}$  and  $\mathcal{R}$  has local energy  $E_{\mathcal{R}} = E - E_{\mathcal{S}}$ . The probability that  $\mathcal{S}$  has energy  $E_{\mathcal{S}}$  for a fixed  $E$  is a conditional probability and is denoted by  $p(E_{\mathcal{S}}|E)$ , i.e., it is the probability of having energy  $E_{\mathcal{S}}$  conditional on the total energy being  $E$ . We can write

$$p(E_{\mathcal{S}}|E)dE_{\mathcal{S}} = \frac{\Omega_{\mathcal{S}}(E_{\mathcal{S}})\Omega_{\mathcal{R}}(E_{\mathcal{R}})dE_{\mathcal{S}}}{Z_E} = \Omega_{\mathcal{S}}(E_{\mathcal{S}})e^{S_{\mathcal{R}}(E_{\mathcal{R}})}dE_{\mathcal{R}}/Z_E, \quad (6)$$

where  $\Omega_{\mathcal{S}}(E_{\mathcal{S}})$  and  $\Omega_{\mathcal{R}}(E_{\mathcal{R}})$  are the number of states of  $\mathcal{S}$  and  $\mathcal{R}$  with energy  $E_{\mathcal{S}}$  and  $E_{\mathcal{R}}$ .  $Z_E$  is the partition function obtained by counting all states with a fixed  $E$

$$Z_E = \int_E dE_{\mathcal{S}} \Omega_{\mathcal{S}}(E_{\mathcal{S}}) e^{S_{\mathcal{R}}(E_{\mathcal{R}})}. \quad (7)$$

We can now expand  $S_{\mathcal{R}}(E_{\mathcal{R}})$  around  $E$  to give  $S_{\mathcal{R}}(E) - E_{\mathcal{S}}\partial S_{\mathcal{R}}(E)/\partial E$ . We then define the inverse temperature  $\beta_E$  in the usual manner in terms of the local extensive entropy

$$\beta_E \equiv \frac{\partial S_{\mathcal{R}}(E_{\mathcal{R}})}{\partial E_{\mathcal{R}}}. \quad (8)$$

We shall refer to  $\beta_E$  as the *local temperature*. The motivation for using this term (as with many of the terms we are introducing) comes from general relativity.

Note that the temperature of the system is defined in terms of the derivative of the *reservoir's entropy*. In the noninteracting case, no issues arise from this definition: if two systems are in thermal contact in the microcanonical ensemble, then  $\partial S_{\mathcal{S}}(E_{\mathcal{S}})/\partial E_{\mathcal{S}} = \partial S_{\mathcal{R}}(E)/\partial E$ . When long-range interactions are present, this is not necessarily true—a point which will be discussed in Sec. II F. One therefore should keep in mind that the temperature is a property of a reservoir—it gives the distribution associated with a smaller system in contact with it. In the case where the division of a *single*

system into a reservoir and smaller system is purely formal, we will see that  $\partial S_{\mathcal{S}}(E_{\mathcal{S}})/\partial E_{\mathcal{S}} = \partial S_{\mathcal{R}}(E)/\partial E$ . This comes from symmetry considerations. Another special case is when the reservoir has no long-range interactions. Both these cases will be discussed in Appendix A. In general, one can relate  $\beta_E$  to  $\partial S_{\mathcal{S}}(E_{\mathcal{S}})/\partial E_{\mathcal{S}}$  using the methods we will develop in Sec. II F.

Using the definition of Eq. (8), we find

$$p(E_{\mathcal{S}}|E)dE_{\mathcal{S}} = \frac{\Omega_{\mathcal{S}}(E_{\mathcal{S}})e^{S_{\mathcal{R}}(E)-E_{\mathcal{S}}\beta_E}dE_{\mathcal{S}}}{Z_E}. \quad (9)$$

One can see from the above expression, that at fixed  $E$ , the probability distribution of  $\mathcal{S}$  looks independent of the nature of the reservoir, since the probability for the system to be at a certain energy  $E_{\mathcal{S}}$  is

$$p(E_{\mathcal{S}}|E)dE_{\mathcal{S}} \propto \Omega_{\mathcal{S}}(E_{\mathcal{S}})e^{-E_{\mathcal{S}}\beta_E}dE_{\mathcal{S}}. \quad (10)$$

which looks like the ordinary canonical ensemble. We must keep in mind, however, that the temperature is defined in terms of the reservoir, not the system.

Now, however, there is a key difference. We have worked in the microlocal ensemble, but in the microcanonical ensemble, it is not  $E$  that is held fixed, but rather  $m$ . If there were no interactions, then this would mean that  $E$  is also fixed (if  $G=0$  then, in fact,  $m=E$ ). Then, since  $E$  would be constant, so would  $E_{\mathcal{R}}(E)$ . We would therefore recover the fact that  $\mathcal{S}$  is in the usual canonical ensemble. However, for  $G(E_{\mathcal{S}}, E_{\mathcal{R}}) \neq 0$ ,  $E$  need not be constant. For example, if  $G(E_{\mathcal{S}}, E_{\mathcal{R}})$  were quadratic in  $E$ , then at fixed total energy  $m$ , the system can be in one of two values of  $E$ . For more complicated potentials, *many values of the local energy  $E$*  are possible even though both  $m$  and  $E_{\mathcal{S}}$  are fixed. Therefore, for an isolated system, one should not hold  $E$  fixed, but rather  $m$  as this is the conserved quantity, while the local energy  $E$  of an isolated system can change.

In order to calculate the probability distribution of the system in the microcanonical ensemble, we can use *the law of total probability*

$$p(y) = \sum_x p(y|x)p(x). \quad (11)$$

To this end, we will use the fact that at fixed  $m$ , the probability that the system has local energy  $E$  is given by

$$p(E) = Z_E/Z_m, \quad (12)$$

where

$$Z_m = \int_m dE_{\mathcal{S}} \Omega_{\mathcal{S}}(E_{\mathcal{S}}) e^{S_{\mathcal{R}}(E_{\mathcal{R}})}, \quad (13)$$

i.e.,  $Z_m$  is the total number of states at fixed total energy  $m$ . Clearly, the probability that the local energy is  $E$  is just given by the total number of states which have local energy  $E$  divided by the total number of states  $Z_m$ .

We therefore find, that in the microcanonical ensemble, the probability distribution of  $\mathcal{S}$  in contact with a large reservoir is given by

$$p(E_S)dE_S = \sum_E \Omega_S(E_S)\Omega_R(E)e^{-E_R\beta_E}dE_S/Z_m, \quad (14)$$

where the sum is taken over all  $E$  consistent with total energy  $m$ . We see that one does not recover the usual thermal distribution. Rather, one has a probability distribution of thermal distributions. There is also no decoupling of the probability distribution of the system from the reservoir. In other words, one does not obtain a simple probability distribution like Eq. (10) which does not depend on the reservoir.

Although the microcanonical distribution is appropriate for an isolated system, there may be situations where the microlocal distribution is also appropriate. Such situations include cases where superselection rules single out a particular  $E$  (for example, charge conservation or angular momentum conservation may not allow transitions from one value of  $E$  to another). One also may have cases where there is a large gap between various values of  $E$  so that once the system takes on a particular value of  $E$  it is unlikely to change, as this would require a large random fluctuation. In such cases, a smaller system in contact with a large one would behave as if it were in a canonical ensemble.

Finally, we note that the local temperature, as we have defined it, is a function of  $E$ . It is for this reason that we have explicitly put in this dependence by writing  $\beta_E$ . There will be different “temperatures” dependent on what value of  $E$  the entire system is found in.

### C. Global temperature and conserved energy

As we saw in the preceding section, a small system interacting with a reservoir behaves as if it is at a number of different local temperatures  $\beta_E$ . Equation (14) gives the probability distribution in terms of this local temperature and the local energy  $E_S$ . However, the local energy is not a conserved quantity, and it is not the energy that an observer will ascribe to  $\mathcal{S}$ , since it does not contain the interacting term. We therefore define the conserved energy of a system interacting with another system  $\mathcal{R}$  as  $E_o \equiv m(E_S, E_R) - m(0, E_R) \approx E_S(\partial m / \partial E_S)$ . This is the change in the conserved energy of  $\mathcal{S}$ , if one only makes changes to its energy levels. Clearly,  $E_o$  is also a function of  $E$  but we will not write this explicitly. We will now show that  $E_o$  gives the energy levels of  $\mathcal{S}$  in the presence of  $\mathcal{R}$ . Using the definition above, we can rewrite Eq. (14) as

$$p(E_S)dE_S = \sum_E \Omega_S(E_S)\Omega_R(E)e^{-E_o\beta_o(m)}dE_S/Z_m, \quad (15)$$

where the global temperature is defined as

$$\begin{aligned} \beta_o &\equiv \frac{\partial S_{\mathcal{R}}(m)}{\partial m} \\ &= \frac{\partial E}{\partial m} \beta_E. \end{aligned} \quad (16)$$

Note that this global temperature does not depend on  $E$ . We have the important relationship

$$\beta_E E_S = \beta_o E_o. \quad (17)$$

$E_o$  can be thought of as the effective energy, i.e., it is the energy of the system in the presence of the reservoir (in general, it is the energy of the system in the presence of its interaction with another system).  $\beta_o$  can be thought of as the closest thing one has to a physical temperature. If this is not yet clear from the above definitions, it should become clearer when we examine the example below.

### D. An example: Physical significance of global quantities $E_o$ and $\beta_o$

We do not have a single thermal distribution, but rather a probability distribution of canonical ensembles, one for each accessible  $E$ . However, the quantity  $\beta_o$  is the same for each ensemble. In Sec. II F, we will see that  $\beta_o$  does possess a crucial quality of temperature—namely, two systems in equilibrium will have the same  $\beta_o$ . Here, we point out some other physical properties that the global quantities  $\beta_o$  and  $E_o$  have.

Let us examine the physical significance of  $E_o$  when we look at the spin model of Eq. (4). Let us imagine that we have a homogeneous system, so that  $h_1 = h_2$  and  $J_1 = J_2 = J_{12}$ , and we can therefore drop the subscript. Let us also work in the microlocal ensemble with fixed  $E$  for the time being as we simply want to understand the significance of  $E_o$  and  $\beta_o$ . Let us consider system 1 to be a single spin (we will henceforth treat system 1 as the small system  $\mathcal{S}$  and system 2 as the reservoir  $\mathcal{R}$ ). This single spin acts like a probe, and can be thought of as a thermometer. Then inserting Eq. (5) into Eq. (9), one can verify that the probability of the thermometer being spin up (and hence, having local energy  $h$ ) is equal to  $(N+e)/2N$  which is exactly as one expects, since this is just the fraction of spins which are up in the entire system (here,  $N$  is the total number of lattice sites). However, the true energy levels of the spin are not  $\pm h$  as they would be if the system was noninteracting. One must also add the field due to all the spins in system 2 (i.e., the rest of the system, not including the probe spin). This means that the single spin actually feels a magnetic field of  $h - Je$ . This is exactly equal to  $E_o$ . Therefore, someone measuring the energy levels of the thermometer would conclude that the thermometer had energy levels  $E_o$  and that based on its probability distribution, it is in a thermal distribution at temperature  $\beta_o$ . This is exactly what is given by Eq. (15).  $\beta_o$  is, therefore, the physically significant temperature from the point of view of this type of a determination of temperature. Note, however, that because of the self-interactions in the system,  $m \neq NE_o$ , i.e., one cannot find the total energy by adding up

all the locally conserved energies of each spin. This is just another manifestation of the nonextensivity of the system.

**E. Physical significance of the local quantities  $E$  and  $\beta_E$**

The temperature of the probe in the above example was  $\beta_o$ , but we will now show how to use a probe to measure the local temperature  $\beta$ . From the point of view of the formalism, the local temperature is a useful quantity. It is clearly an intensive quantity, as can be seen from Eq. (2). On the other hand, the global temperature is not an intensive quantity. Doubling the size of the system along with the total energy will result in a change in the temperature, as can be seen simply from its definition (we will see encounter this more explicitly in Sec. II F). We will also find that the local temperature has a very strong physical significance in general relativity, where it is the physical temperature measurable by free-falling observers. One would therefore like to know how to measure it in other theories. It can be measured using the following method. We again use the example from the preceding section of a single spin probe interacting with a larger system and identical to it.

We use the probe to measure the temperature by first slowly and adiabatically drawing the probe away from the rest of the spins until it no longer interacts with the system (i.e.,  $J_{12}=0$ ). Its energy levels then become  $\pm h$ . From Eq. (17), one sees that this spin then acts like a thermometer measuring the local temperature  $\beta_E$ , i.e., in an adiabatic change the state of the system remains constant and its average spin is still  $e/N$ . However, the energy levels we would ascribe to the spin are now different. We would no longer say that the energy levels are given by  $E_o$  since the magnetic field is now zero and the  $eJ$  term no longer contributes. As a result, the spin acts as if it is in a thermal bath of temperature  $\beta_E$  and has energy levels  $\pm h$ .

One can use an almost identical method which reminds one of the measurement made by a freely falling observer in general relativity. In general relativity, the local temperature is in fact the physical temperature as measured by an individual who is freely falling. The global temperature  $\beta_o$  is the temperature measured by an individual at infinity. With a spin system, there is no distinction between different observers. However, an analogy with general relativity motivates the following method. One slowly (adiabatically) applies a local magnetic field  $B=Je$  to the probe spin. This is exactly the magnetic field which cancels the magnetic field of the surrounding spins which are acting on the probe. The probe spin then has energy levels of  $\pm h$  and since its state has not changed, it is as if it is in a heat bath of temperature  $\beta_E$ . One can think of the applied field as being analogous to the gravitational “force” which gets canceled when one goes into free fall.

It is worth noting that there is a significant difference between the method for measuring the local temperature in the case we have described and the case where the interactions are short range (in the Ising model say). For example, when we remove a single spin from this system, we are moving it in a known potential, since the interaction is mostly due to the entire system as a whole. On the other

hand, in the Ising model, where the potential is due to nearest neighbor interactions, one does not know the local potential, since it is random. Therefore, in the long-range case, one can liberate all the long-range interaction energy by removing a spin. In the Ising model, much of the interaction energy is contained in thermal fluctuations and all of it cannot be liberated in such a manner.

**F. Unequal local temperatures at equilibrium, equal global temperature**

It is a standard result that for two noninteracting systems, the temperatures will be equal when they are brought into equilibrium. We now extend these results by showing that the local temperature of two systems in thermal contact, need not be equal, while each system’s global temperature will be equal. This in many respects justifies calling  $\beta_o$  a temperature, even though it is not the temperature in the usual sense, since the microcanonical ensemble does not lead to a canonical ensemble.

We allow the two subsystems to exchange energy but keep the total energy  $m$  fixed, while the local energy  $E$  need not be fixed. At equilibrium, the system will be found in the most probable configuration, i.e., the entropy will be an extremum so that the system is in the macroscopic state with the maximum number of microstates. We can then find the extremum by varying  $E_1$  and  $E_2$  at fixed  $m$ . The entropy of two systems is given by Eq. (2), and we now find the extremum to give the most probable configuration

$$dS = \left( \frac{\partial S_1}{\partial E_1} \right)_m dE_1 + \left( \frac{\partial S_2}{\partial E_2} \right)_m dE_2 = \left[ \frac{\partial S_1}{\partial E_1} \frac{\partial E_1}{\partial E_2} + \frac{\partial S_2}{\partial E_2} \right]_m dE_2 = 0. \tag{18}$$

Using the definition of local temperature of Eq. (8), we find

$$\beta_2 = -\beta_1 \left( \frac{\partial E_2}{\partial E_1} \right)_m = \beta_1 \left[ 1 + \left( \frac{\partial G(E_1, E_2)}{\partial E_1} \right)_m \right], \tag{19}$$

where  $\beta_i$  is the local temperature of each system, and the dependence on  $E$  is implied. One can also easily verify that the global temperatures of each system are equal, just by using Eq. (19) and the definition of  $\beta_o$ . This to an extent justifies the term *temperature*. Note, however, that if we move two systems together which are both initially at the same global temperature  $\beta_o$ , then their new equilibrium temperature can be at a different global temperature  $\beta'_o$ . This is due to the addition of new coupling terms in the total energy. The global temperature is therefore not an intensive quantity. We will see this in the example below.

Using the two coupled Ising models of Eq. (4), we would get a temperature difference of

$$\beta_2 = \beta_1 \frac{1 - J_{12}e_1/h_2 - J_2e_2/h_2}{1 - J_{12}e_2/h_1 - J_1e_1/h_1}. \quad (20)$$

One can solve for  $e_1$  and  $e_2$  as a function of the total spin excess  $e$  by explicitly calculating the temperature using Eq. (5) and equating it with the equation above. Solving these two equations for  $e_1(e_2)$ , will then allow us to give an expression for the temperature difference.

$$\frac{\partial S_1/\partial E_1}{\partial S_2/\partial E_2} = \frac{1 - J_{12}e_1/h_2 - J_2e_2/h_2}{1 - J_{12}e_2/h_1 - J_1e_1/h_1}. \quad (21)$$

Using Eq. (5) for the entropy of the  $i$ th system gives

$$\frac{\partial S_i}{\partial E_i} = \frac{1}{2h_i} \ln \frac{N_i - e_i}{N_i + e_i}. \quad (22)$$

This gives the following equation:

$$\left( \frac{N_1 - e_1}{N_1 + e_1} \right)^{h_2^2(h_1 - J_{12}e_2 - J_1e_1)} = \left( \frac{N_2 - e_2}{N_2 + e_2} \right)^{h_1^2(h_2 - J_{12}e_1 - J_2e_2)}, \quad (23)$$

which can be solved graphically for  $e_1(e_2)$ . This can then be substituted back into Eq. (20). We shall not do so here.

Now if initially these two systems (or clusters), are far apart, and at equal global temperature, then when pushing them together one cannot do so both adiabatically and isothermally (constant global temperature) as one can do in the noninteracting case. This can be seen from Eq. (17). Moving the systems together adiabatically requires keeping  $E_o\beta_o$  fixed. But since  $E_o$  changes when  $J_{12}$  becomes significant, one cannot keep  $\beta_o$  constant. By recalculating  $E_o$ , one can therefore calculate the new global temperature. We see therefore that the global temperature is not an intensive quantity.

Finally, one can consider what happens when one is in the grand-canonical ensemble, i.e., we allow the number of particles  $N_1$  and  $N_2$  to change, while keeping the total number of particles  $N$  and volume  $V$  fixed. In this case, one can define the local chemical potential in the same way as we defined the local temperature

$$\mu = -T \left( \frac{\partial S}{\partial N} \right)_{E,V}, \quad (24)$$

and one finds that

$$\mu_1\beta_1 = \mu_2\beta_2. \quad (25)$$

This leads one to see that the local chemical potentials of two systems will also not be equal, and that the ratio between the two chemical potentials is the inverse of Eq. (19). One can likewise define a global chemical potential

$$\mu_o = \frac{\partial m}{\partial E} \mu_E. \quad (26)$$

Finally, the preceding discussion allows us to write the average entropy of two systems. Previously, we wrote the entropy of two systems as  $S(E_1, E_2)$ , as in Eq. (2), i.e., we

gave the entropy of the two systems when one had energy  $E_1$  and the other  $E_2$ . However, if the two systems are in contact, the energies will fluctuate until the system is in its most probable configuration. We can therefore write the average entropy of the two systems as

$$S(E) = \bar{S}(E_1, E_2), \quad (27)$$

where it is understood that this is the average entropy of the combined system.

### III. AN EXAMPLE: THE LONG-RANGE LATTICE MODEL

In order to better understand the points raised in the previous sections, it will be useful to work out a very simple example. We will consider a single system of spins interacting via a long-range potential which is constant. This model is simpler than the one of Eq. (4), and has total energy

$$m = he - Je^2/2, \quad (28)$$

where we have simply dropped the subscripts from Eq. (4) and have got rid of the second cluster.

We essentially work in the microcanonical ensemble and use the formalism we have introduced. There are other long-range lattice models which attempt to solve similar interactions such as the Curie-Weiss model, where the interaction is made to scale inversely to the number of lattice sites. The dependence of the interaction on the size of the system is problematic, but it ensures that the thermodynamic limit exists. A generalization of this is the Kac model [15] which has an interaction with an exponential cutoff. Here, because of our formalism, there is no need to introduce such a cutoff. The model we explore is likewise related to the mean field approximation of the Ising model, although in the mean field approximation, it is as if one is working in the microlocal ensemble, rather than the microcanonical ensemble as we do here. In other words, in the mean field approximation, one will not see the effects of having multiple values of  $E$ . Another model which has been extensively studied are those with  $1/r$  potentials (see Ref. [16] for a recent review).

In Eq. (28) the local energy is  $E = he$ , and for fixed  $m$  there are two possible values of the local energy which can be obtained by solving Eq. (28) for  $e$

$$e_{\pm}(m) = \frac{h}{J} [1 \pm k(m)], \quad (29)$$

where  $k$  is given by

$$k = \sqrt{1 - \frac{2Jm}{h^2}}. \quad (30)$$

It is these two values of  $e$  which will give us the two different local temperatures. From Eqs. (16) and (28), we get the relationship between the global and local temperatures

$$\begin{aligned} \beta(e_{\pm}) &= \beta_o \left( 1 - \frac{J e_{\pm}}{h} \right) \\ &= \mp \beta_o \sqrt{1 - \frac{2mJ}{h^2}}, \end{aligned} \quad (31)$$

and likewise, using  $E_o = E \partial m / \partial E$ , we get

$$\begin{aligned} E_o &= (1 - J e_{\pm} / h) h e_{\pm} \\ &= \mp k(m) E. \end{aligned} \quad (32)$$

We will henceforth use  $\beta_{\pm}$  to represent the two local temperatures. It is also worth noting that Eq. (32) for the conserved energy  $E_o$ , gives the energy levels of a spin in the presence of the effective magnetic field due to the applied field  $h$  and all the other spins of the system. In other words,  $E$  is the raw energy levels  $\pm h$  in the absence of the long-range interactions, while  $E_o$  gives the energy levels in the presence of interactions.

One can also arrive at Eq. (31) using the following method which is highly illustrative. Consider the example in the preceding section of two spin systems with a local temperature difference given by Eq. (20). Now imagine that system 2 is being used as a thermometer to measure the temperature of system 1, i.e., system 2 has no long-range interactions and minimal energy  $J_2 = J_{12} = 0$ . In this case, Eq. (20) gives

$$\beta_2 = \frac{\beta_1}{1 - J_1 e_1 / h_1}. \quad (33)$$

This is precisely the same relation as Eq. (31) giving the relationship between the local and global temperatures. We can conclude from this that a spin which does not have long-range interactions with the rest of the system will “feel” the global temperature—its local temperature will be the system’s global temperature. On the other hand, a system which is identical to the rest of the system will obviously have the same local temperature as the rest of the system.

However, as we know this temperature depends on which branch of local energy the system is in, i.e., whether it is in the state  $e_+$  or  $e_-$ . If one were to look at a single spin in order to determine the temperature, one would not find the spin in a thermal state. Rather, the spin would be in a distribution given by Eq. (15), i.e., it is in a distribution of two possible canonical ensembles, with two local temperatures  $\beta_{\pm}$  corresponding to spin excesses of  $e_{\pm}$ . The conditional probability of a single spin being up, given a spin excess of either  $e_{\pm}$  is given by Eq. (9), i.e., the probability of it being spin up having local energy  $h$  is

$$p(h|e_{\pm}) = \frac{e^{\beta_{\pm} h}}{e^{\beta_{\pm} h} + e^{-\beta_{\pm} h}}. \quad (34)$$

The probability of the system being in the state  $e_+$  or  $e_-$  is given by Eq. (12)

$$p(e_{\pm}) = \frac{\Omega_{\pm} \cosh(\beta_{\pm} h)}{\Omega_- \cosh(\beta_- h) + \Omega_+ \cosh(\beta_+ h)}, \quad (35)$$

where  $\Omega_{\pm}$  is the number of states of the total system with spin excess  $e_{\pm}$ . It is given by

$$\Omega_{\pm} = \frac{N!}{\left(\frac{N+e_{\pm}}{2}\right)! \left(\frac{N-e_{\pm}}{2}\right)!}, \quad (36)$$

and can be approximated by Stirling’s equation. The total probability of a spin being up or down is then given by Eq. (15),

$$p(\pm h) = \frac{\Omega_+ e^{\pm \beta_+ h} + \Omega_- e^{\pm \beta_- h}}{2\Omega_- \cosh(\beta_- h) + 2\Omega_+ \cosh(\beta_+ h)}. \quad (37)$$

There is nothing special about the particular spin we are using as a probe, and so the average orientation  $p(h) - p(-h)$  should be equal to the average magnetization of the entire system  $e/N$ . We therefore have

$$e(\beta_o) = N \frac{\Omega_- \sinh(\beta_- h) + \Omega_+ \sinh(\beta_+ h)}{\Omega_- \cosh(\beta_- h) + \Omega_+ \cosh(\beta_+ h)}, \quad (38)$$

while in the microlocal ensemble (fixed  $e_{\pm}$ ), we have

$$e_{\pm}(\beta_o) = N \tanh(\beta_{\pm} h). \quad (39)$$

Finally, we note that one finds the usual phase transitions at  $\beta_o = J/2$ . Here, however, the phase transition is real, not like the false phase transition that can occur in the mean field approximation. It therefore exists in one dimension. It will also have the property that there will be different average  $e$ , depending on which microlocal ensemble the system is in.

## IV. A BLACK HOLE ANALOG

### Lattice model as a black hole analog

In general relativity, one encounters a number of interesting thermodynamical effects. Perhaps the most well known is the Tolman relation [10]. One finds, that in curved space, the temperature, as measured by local free-falling observers varies from point to point. This is usually interpreted as being due to the redshifting of frequencies due to the curvature of space-time. We have already seen that for other systems with long-range interactions, one has a variation of the local temperature. The results are closely related, and here we will see that they can have exactly the same form. This gives a new interpretation to the Tolman relation; it is not the sole domain of general relativity as is usually believed, but instead arises in other theories with long-range interactions. However, what makes the variation of local temperature special in general relativity, is the physical meaning it has. We saw that in the lattice model, the local temperature was not a constant throughout the system (just as in general relativity)—however, the local temperature of the lattice did not have the same physical interpretation as it does in general relativity (where it is the actual temperature as far as



free-falling observers are concerned). In the lattice model, the local temperature was only the physical temperature if we applied a local magnetic field to cancel the effective magnetic field due to the rest of the system. This is closely related, however, since the applied magnetic field is very similar to the effect of free fall in general relativity.

Another important thermodynamical phenomenon in general relativity is the black hole entropy. This is often viewed as the key to understanding quantum gravity, since any theory of quantum gravity should presumably predict the correct value of the black hole entropy. It is therefore important to understand what aspects of the black hole entropy are specifically related to gravity, and what aspects arise in other theories. We are therefore motivated to explore the similarities of our lattice model, with the types of effects one finds in general relativity.

Inspecting Eq. (31) for the lattice model, one cannot help but be struck by its similarity with the Tolman relation from general relativity. Indeed, defining  $r=h^2$  and putting the coupling  $J=G$ , where  $G$  is the gravitational coupling constant, we see that Eq. (31) becomes

$$\beta_{\pm} = \mp \beta_o \sqrt{1 - \frac{2Gm}{r}}. \quad (40)$$

The positive solution is exactly the Tolman relation for the redshifting of temperature in the Schwarzschild geometry—the Schwarzschild geometry being the space-time of an uncharged nonrotating black hole or spherically symmetric star. In Appendix B, we will see that one obtains similar redshifting effects in a theory with a Newtonian potential.

Let us now show that our Ising model can be thought of as a black hole analog. For  $r > 2Gm$ , Eq. (40) behaves like the exterior Schwarzschild solution (the solution outside of the black hole). As we decrease  $r$ , the local temperature becomes hotter and hotter (for fixed global temperature).

Setting  $r=2Gm$  gives the black hole analog (or perhaps more appropriately, the point in space where an observer would be on the horizon). In this case, there is an infinite “redshifting” between the global temperature and the local temperature as can be seen from Eq. (40). This “black hole” solution is not only special because of the divergence in the redshift—it is also the point where the two solutions  $e_+$  and  $e_-$  coincide. There is therefore only one local temperature. One can therefore see that it is only the black hole analog solution which is thermal. All other solutions do not give a thermal distribution. The black hole solution is also special in that there is a degeneracy in the energy levels—a point which will become important when we discuss the system’s entropy.

There are two interesting cases to consider: (1) the case where the global temperature  $\beta_o$  is finite and (2) where the global temperature is zero.

In case (1), if the global temperature is finite, then at the point that  $r=2Gm$  the local temperature diverges. This is explained in the spin model by inspecting Eq. (32) and noting that the point  $h^2=2Jm$  (the analog of the black hole horizon) corresponds to  $h=Je$ . This is exactly the point  $E_o=0$  and therefore an individual spin sees no effective mag-

netic field. In other words, both the energy levels of a single spin are zero. Therefore, since  $\beta E = \beta_o E_o$  is finite for finite  $\beta_o$ , we must have an infinite local temperature. Also, since individual spins see no net magnetic field, there is no preferred spin direction, and one finds  $e=0$ . This solution is therefore the solution with maximal entropy.

Case (2), with zero global temperature, can be thought of as being analogous to the extremal black hole (i.e., the charged black hole solution with zero temperature). In this case, we can have a finite local temperature. This again comes from the relation  $\beta E = \beta_o E_o$  and the fact that  $E_o=0$ . In this case,  $e$  is more or less arbitrary.

The gravitational analog has another interesting property. In gravity, one requires that the radius of the black hole is at  $R=2Gm$  which is the so-called Schwarzschild radius, one cannot have  $R < 2Gm$ . The same is true here. There is no value of  $e$  which would allow  $h^2 < 2Jm$ . The black hole analog solution occurs at the maximum of energy  $m = h^2/2J$  which is exactly the Schwarzschild radius. The limit  $h^2=2Jm$  therefore corresponds to a horizon in a very real sense.

There is also an analog of the “white-hole” solution, which occurs when  $J$  is taken to be negative. In general relativity, the white hole is a solution to Einstein’s equations, however, it is unphysical as it is unstable, and entropically forbidden, requiring highly special initial conditions. However, with the spin model, such a solution is completely physical (and also more stable). It corresponds to a minimum of energy  $m$ .

The case of  $e_-$  reminds one of the situation interior to a black hole, since in this case, the local temperature is negative as can be seen from Eq. (40). Likewise, inside the black hole, the light cone is tilted over, in such a way that energies which are positive outside the hole, are negative inside. In the black hole analog case, the negative local temperature can be understood from Eq. (32) by noting that positive conserved energies  $E_o$  correspond to negative local energies. In other words, a spin is more likely to be pointing up with energy  $h$ , even though this corresponds to a greater local energy (it instead corresponds to a smaller conserved energy).

We now turn our attention to the entropy of these solutions. This is of interest because the black hole entropy is usually considered to have some unique properties.

(1) The black-hole’s entropy is proportional to its area and not its volume. In the case of three spatial dimensions, this means that the black hole entropy is proportional to the square of its total energy (since the black hole radius is at  $2m$ ).

(2) The black-hole entropy is universal—the constant of proportionality between the entropy and the area is the same regardless of the past history of the black hole. This means that no matter what type of initial matter formed the black hole, its final entropy will only depend on the total energy of the black hole (or other conserved quantities in the case of charged and rotating black holes).

(3) Before a system forms a black hole, its gravitational entropy is zero, while the black hole entropy (which is enormous) appears suddenly, when the system forms a black hole. The system may have some material entropy before it

forms the black hole, but this is negligible compared to the sudden increase in entropy it gets when it forms a black hole. The particular spin model under discussion does not possess properties which are identical to the black hole. However, it does possess characteristics similar to the three mentioned above.

To aid in this discussion, it is worthwhile to add the constant terms back into our expression for the total energy  $m$  of the spin model. From Eq. (3) one can put back the constants, so that instead of Eq. (28), one gets

$$m = he^2 - Je^2/2 - JN/2, \quad (41)$$

where  $N$  is the total number of spins.

Now the entropy one finds will depend on what one specifies about the system. Consider the case when one knows not only  $m$ , but also the energy  $E_o$  of every spin. In this case, the entropy of the system is zero, since knowing the energy  $E_o$  of each spin is the same as knowing the spin itself, so one has complete knowledge of the system. However, when the system becomes a black hole analog at  $h^2 = 2Jm$ , the energy levels  $E_o$  of each spin become doubly degenerate and the system suddenly acquires an entropy of  $\log 2$  per spin. This then is similar to property (3) of a black hole. Such a property also exists if one does not know the energy of each spin, but instead knows the total local energy of the system, i.e., one knows whether the system is in the  $e_+$  state or in the  $e_-$  state. When the system forms the black hole analog, these two different states merge, and one acquires an additional (although negligible) entropy of  $\log 2$ . However, one interesting property of this entropy  $\log 2$  is that it is only a function of the form of the interaction, and not of the particular system. In this system, we have a factor of  $\log 2$  because the potential is quadratic, and there are two possible local energies  $he_{\pm}$  for fixed total energy  $m$ . It is tempting to regard this as a type of universality, similar to property (2) of the black hole. The factor of  $\log 2$  comes because of the form of the interaction, and has nothing to do with the particular system, just as the black hole entropy comes from the gravitational interaction and has nothing to do with the particular system. For a general potential, there will be  $n$  possible local energies at fixed energy and one might regard  $\log n$  as being the entropy associated with the interaction. One can also make the degeneracy arbitrarily large, by considering higher level spins, rather than just the two level systems we have been considering here.

Note that the degeneracy in  $E_o$  which occurs in the spin model also has a counterpart in the black hole. There, one also finds that the conserved energy is zero on the horizon. In some sense, this is what enables one to “pack” a large amount of entropy at no energy cost, close to the horizon.

Finally, one can ask about the nonextensive properties of the entropy. The entropy is nonextensive in the sense that it does not scale proportionally with the total energy  $m$ . In other words, the entropy is proportional to the local energy  $E$  (and  $N$ ), but because the total energy  $m$  does not scale linearly with  $E$  and  $N$ , the entropy will not be a linear function of the total energy.

If one uses the entropy as given by Eq. (5) then the entropy will scale as both  $N$  and  $e$ . If one uses the definition of entropy discussed above, then the entropy will scale with  $N$ . On the other hand, the scaling of the total energy is given by Eq. (41) so a general system will not have an entropy which scales like  $m$ . For the case of the black hole solution, i.e.,  $h^2 = 2Jm$ , the total energy is

$$m = \frac{J(e^2 - N)}{2}, \quad (42)$$

which has a completely nonextensive part (scaling like  $e^2$ ) as well as the extensive part (scaling like  $N$ ).

Here, the entropy, while exhibiting nonextensivity as a function of total energy, does not scale the same way as a black hole (i.e.,  $S \propto m^2$ ). However, one can imagine easily constructing an interaction which has an entropy which has the same dependence on  $m$  as the black hole. For example, by having an interaction of the form  $m \propto \sqrt{E}$ . Then, for a locally extensive system (i.e.,  $S \propto E$ ) such as the spin models we have been considering, one will find the same entropy scaling behavior as a black hole.

We will discuss in more detail in Sec. V how one can derive the scaling relations for the entropy based on the considerations introduced here.

Finally, we note that one can arrange the phase transition so that it puts the system at  $e = 0$  and one finds a second-order phase transition into the black hole analog solution. This is in contrast to the black hole case, where the jump in entropy suggests a first-order phase transition. However, if one looks at the entropy given that one knows the value of  $E_o$  for each spin, then the analog does indeed have a discontinuity in entropy.

## V. NONEXTENSIVE SCALING LAWS

As discussed in Sec. IV, the scaling of the entropy will no longer be purely extensive. Here, we will show how to quantify the degree of nonextensivity for particular systems. The main idea is to use the principle of local extensivity which is given as Eq. (2). In other words, in terms of the local energy  $E$ , the entropy is an extensive quantity. This can be written as

$$S(\lambda E) = \lambda S(E) \quad (43)$$

in terms of the total energy  $m$ , the system will not be extensive. Here, we will work in the microlocal ensemble—in the end, one must sum over all  $E$  consistent with  $m$ . We will therefore in this section write  $S$  as a function of  $E$  to remind ourselves of this. In the case of densities, it is understood that  $s$  is a function of  $\rho$ .

We now use a second ingredient, namely, the Gibbs-Duhem relation [17]

$$s = \beta(\rho + p) - \mu n \quad (44)$$

to relate the various thermodynamical quantities to each other. Here,  $s$ ,  $\rho$ ,  $n$  are the entropy density, energy density, and particle number density and  $T$ ,  $\mu$  and  $p$  are the local temperature, chemical potential, and pressure.

It will prove easier (although not necessary), to multiply Eq. (44) by a tiny volume element  $V$  to get

$$E = TS(E) - pV + \mu N. \quad (45)$$

The standard derivation of the Gibbs-Duhem relation follows from the first law and the principle of local extensivity. One has

$$dE = TdS - pdV - \mu dN. \quad (46)$$

Then, analogously to Eq. (43), one has

$$N(\lambda E) = \lambda N(E), \quad (47)$$

and also that in a small volume, the quantities  $T$ ,  $p$ , and  $\mu$  are intensive, i.e., they do not change with  $\lambda$ . One can then integrate the first law to obtain the Gibbs-Duhem relation. This relation is known as an Euler relation of homogeneity 1. Quantities which scale like  $\lambda^a$  are Euler relations of homogeneity  $a$ .

We can now express the local energy  $E$  as a function of total energy  $m$ , and also use the expressions, Eqs. (16) and (25), to express the local temperature and chemical potential in terms of their global quantities  $\beta_o$  and  $\mu_o$ . Or, in the case of a continuum system, one can use Eqs. (C4), (C7), and (C9) We would thus have all local quantities expressed in terms of global ones.

In the continuum case, one then gets for the entropy density

$$s(\rho) = \beta_o \frac{\partial \rho}{\partial m} (\rho - p) - \mu_o \beta_o n, \quad (48)$$

which can then be integrated to give the total entropy in terms of global quantities. Just as we used  $E_o$ , we here use the conserved energy density  $\rho_o$ . The equation for  $p(x)$  will depend on the the potential. In the following section, we will calculate this quantity for a gravitating perfect fluid, and we will see that the entropy will not scale like the volume of the system, but rather, approaches area scaling behavior as the system becomes more strongly interacting.

As a general rule, one will obtain  $S(m, \beta_o)$ . From this, one can then calculate  $S(\lambda m, \beta_o)$  in order to determine the scaling behavior of the entropy. For general interactions, one find that  $S(\lambda m, \beta_o) \neq \lambda S(m, \beta_o)$ . Instead, for homogeneous potentials, one finds

$$S(\lambda m, \beta_o) = \lambda^a S(m, \beta_o), \quad (49)$$

and the exponent  $a$  then quantifies the degree of nonextensivity of the system.

Perhaps the most famous example of this, is the case of a black hole in three spatial dimensions, where, one finds the so-called Smarr relation [18],

$$S = \beta_o m/2, \quad (50)$$

which is an Euler relation of homogeneity 2 in contrast to the noninteracting case of  $S = \beta E$  which is an Euler relation of homogeneity 1.

One can show that the Smarr relation quantifies the non-extensive nature of the interaction. Differentiating the Smarr relation Eq. (50) and applying the first law  $dm = T_o dS$ , one obtains (in terms of a constant  $\gamma$ )

$$S = \gamma m^2, \quad (51)$$

so that the entropy scales not as  $m$  as it would in the noninteracting case, but as  $m^2$ . We had such a term in our long-range lattice model. The black hole's radius is at  $R = 2m$  and so in terms of the black hole area  $A$ , one obtains

$$S = \frac{\gamma}{16\pi} A. \quad (52)$$

One likewise gets

$$T_o = (2\gamma m)^{-1}, \quad (53)$$

which is the correct expression (up to a constant of proportionality) for the Bekenstein-Hawking temperature.

Finally, it is worthwhile to explore some additional relationships one gets for extensive systems. Taking the derivative of the Gibbs-Duhem relation Eq. (45), and applying the first law Eq. (46), gives

$$dT S = V dp - N d\mu, \quad (54)$$

which yields the following two relationships:

$$\left( \frac{\partial T}{\partial p} \right)_\mu = \frac{V}{S}, \quad \left( \frac{\partial T}{\partial \mu} \right)_p = \frac{N}{S}. \quad (55)$$

These relationships between intensive and extensive variables only hold for extensive systems, although they hold locally for nonextensive systems.

## VI. ENTROPY SCALING BEHAVIOR IN GENERAL RELATIVITY

General relativity is another theory in which our assumptions of locality and local-extensivity hold. What's more, quantities like the local temperature have a very real physical meaning—the local temperature is the physical temperature measured by an observer in free fall. We will first discuss how our two assumptions hold in general relativity. Then, we will discuss the entropic scaling relations for the perfect fluid. The principle motivation for the latter study comes from Ref. [13], where it is shown that the entropy of a spherically symmetric material (approximated as a densely packed set of shells) has an entropy which is area scaling at the point before it forms a black hole. It is therefore seen that the area-scaling property of entropy is not unique to the black hole. This suggests that this property arises from the long-range interactions of gravity, and is not solely due to the horizon. Here, we will see similar behavior, however, because we can solve the equations exactly, we can trace the entropy scaling behavior at all values of the gravitational coupling constant.

Let us first see how our two assumptions hold in general relativity. A review of thermodynamics in curved space can

be found in Ref. [19]. Let us first consider the case where there is no gravitational interactions. The thermodynamical quantities,  $\rho$ ,  $n$ ,  $T$ ,  $\mu$ ,  $p$ , and  $s$  are taken to be the quantities measured in the rest frame of the substance. Let us now consider the case where we have gravitational interactions. In this case, we can go into the proper rest frame of the material and consider an observer who is released into free fall. By the principle of equivalence, this observer would measure the same quantities  $\rho$ ,  $n$ ,  $T$ ,  $\mu$ ,  $p$ , and  $s$  (these are what we called the local variables). An equation like the Gibbs-Duhem relation of Eq. (44) is a scalar equation. Since it holds in the nongravitating case, it also holds for the local free-falling observer. Furthermore, since it is a scalar equation, it holds for all observers.

In general relativity, the local temperature  $T = (\partial s / \partial \rho)^{-1}$  is a very real quantity, as it is the temperature as measured by local free-falling observers. Likewise, the global temperature,  $T_o$  is the temperature that would be measured by an observer at infinity. This corresponds almost exactly to the case we were considering in the long-range lattice model. There, the global temperature could be measured by isothermally taking a spin and moving it away from the system so that it no longer felt the interaction, and then measuring its temperature (this is like measuring the temperature at infinity). The local temperature could be measured by canceling out the local magnetic field caused by the interaction, just as going into free fall causes one to not feel the gravitational “force” (not including the tidal force). Although in general relativity the local temperature is just as “real” as the global temperature, this cannot be used to create a perpetual motion machine, because the energy one could extract by moving from a hot local temperature to the cold temperature at infinity is exactly canceled by the work needed to escape the gravitational potential. Gravity is universal, i.e., all objects feel it, so there is no heat engine that could be used to create a perpetual motion. In contrast, not all heat engines would feel the spin-spin interaction which we introduced in the lattice model, however, there, the local temperature did not have the same physical meaning as it does in general relativity. This is because the energy levels of each spin are best described by the conserved energy  $E_o$  and not by the local energy  $E$ . It is interesting that one requires the equivalence principle in order for the local temperature to be a real physical temperature. On the other hand, if the local temperature is physical, then one needs the universality of gravity in order to protect the second law of thermodynamics.

We now turn to the entropy scaling behavior of the gravitating fluid. We use the Gibbs-Duhem relation to calculate the scaling behavior. The actual calculation, while instructive, is done in Appendix D. We also describe how to perform such calculations in greater generality in Appendix C.

A related calculation to the one here is that of Zurek and Page [20] who have calculated the entropy (numerically) for the case of a perfect fluid surrounding a black hole, assuming a specific equations of state.

For a spherically symmetric fluid of constant density, one can calculate the entropy exactly, and it is given by

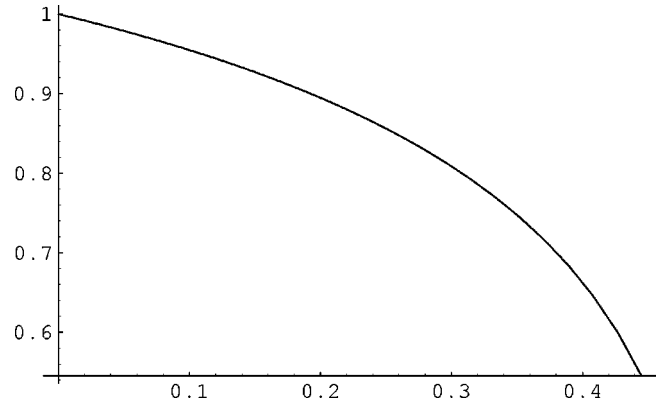


FIG. 1.  $\gamma$  vs  $m/R$  for a perfect fluid of constant density.

$$S = \frac{3k(R)R\beta_o}{4} \left( 1 - \frac{N\mu R}{m} \right) \left[ \sqrt{\frac{R}{2m}} \arcsin \sqrt{\frac{2m}{R}} - k(R) \right], \quad (56)$$

where  $R$  is the radius of the fluid and  $k$  is given by

$$k(R) = \sqrt{1 - 2m/R}, \quad (57)$$

and is virtually identical to Eq. (30).  $m$  is the total energy of the material as measured at infinity [the Arnowit-Deser-Misner mass (ADM) [21]], and therefore we use the same symbol  $m$  we have been using for the total energy. Likewise,  $\beta_o$  is the temperature as measured at infinity, and is thus the same quantity as we have been calling the global temperature. Here, the gravitational constant  $G$  has been set to 1.

Earlier, we saw that for a black hole, we had the Smarr relation  $m = 2T_o S$ , while for ordinary matter  $m = TS$ . We showed that the factor of 2 yielded the area-scaling property of the black hole. It is therefore interesting to see how the entropy of the perfect fluid behaves. Indeed, putting the chemical potential to 0, we can calculate  $\gamma \equiv ST_o/m$  as given by Eq. (57). This is done in Fig. 1. We essentially plot  $\gamma$  versus the strength of the gravitational interaction  $m/R$ . We could have also put back the constant  $G$  (in which case one has  $m/R \rightarrow Gm/R$ ) and plotted  $\gamma$  versus  $G$  holding  $m/R$  constant.

We find, that when the gravitational interaction is weak (i.e.,  $m/R$  is small), the quantity  $\gamma$  is 1 just like in ordinary matter. As we increase the strength of the gravitational field,  $\gamma$  gets smaller, and approaches 1/2 just as it would for a black hole. However, we cannot plot  $m/R$  greater than 4/9, since at this point, the central pressure diverges. The strength of the interaction which corresponds to a black hole is  $m/R = 1/2$  (the Schwarzschild radius). This can, however, be obtained if we have not only central pressure, but also tangential pressure. Indeed, we have done this for spherical shells which have such tangential pressure, and seen that the matter becomes area scaling before a black hole forms [13]. We see therefore that while the system obeys the Gibbs-Duhem locally, it does not obey it globally. This suggests that the fact that black holes have an entropy proportional to their area may be related to the long-range interactions of gravity

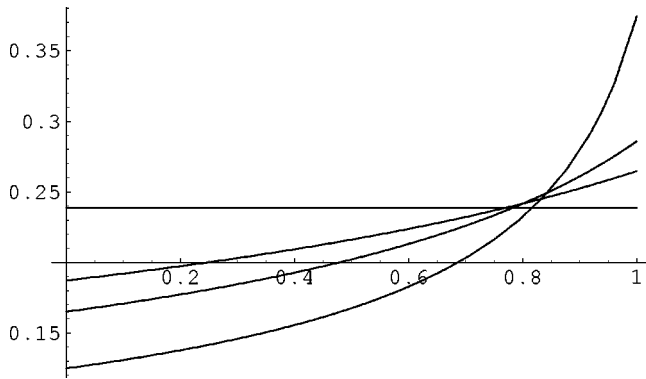


FIG. 2. Fraction of entropy density  $s(r)/S$  vs radius (normalized to 1) for the perfect fluid. The curves plotted are for a strength of gravitational interaction  $m/R=4/9$ ,  $1/3, 1/4$ , and 0.  $m/R=0$  corresponds to the straight line, while  $m/R=4/9$  corresponds to the uppermost line.

rather than only being a special property of the horizon. Area scaling in gravitational systems exists even though there is no black hole horizon.

The distribution of entropy is plotted in Fig. 2 for various strengths of interaction. When the gravitational interaction is negligible, the entropy is constant throughout the sphere as one expects. As the strength of the gravitational coupling is increased, the entropy moves to the surface of the sphere. This intriguing effect helps in explaining why the entropy becomes more area scaling. In the case of tangential pressure, where one can actually approach the black hole radius, one finds that all the entropy lies at the surface of the material.

There is another remarkable property of the entropy of a perfect fluid which is worth mentioning. One should ask whether the entropy as we have calculated, is an extremum. Indeed, it is, however, only an extremum if Einstein's equations are satisfied [22,23]. It is remarkable, because there seems to be no *a priori* reason why the entropy should only be an extremum in curved space for the particular space-time given by Einstein's equations. This interesting connection is discussed in some detail in Ref. [23].

## VII. CONCLUSION

We have introduced a formalism for studying the thermodynamics of interacting systems. The formalism is partly inspired from our understanding of thermodynamics in general relativity. This allows us, not only to use general relativity to understand nonextensive thermodynamics, but also one can learn more about thermodynamics in curved space by looking at thermodynamics in other interacting theories.

We have seen, for example, that many of the properties of black hole entropy also exist in other systems. Likewise, the redshifting of temperatures has a place in other theories of gravity which are not metric theories and have a flat space-time. One can therefore conclude that many of the effects in general relativity have an analog in more classical theories. These results are helpful when attempting to construct a quantum theory of gravity because it enables one to separate

the accidental aspects of black hole thermodynamics from the more fundamental ones.

We have also seen several new effects in nonextensive systems which are worthy of more exploration. We have found that the local temperature can vary throughout a substance, and also that an isolated system can appear to be in a distribution of different temperatures, i.e., canonical ensembles. It would be interesting to apply this formalism to other theories.

Here, we have studied fairly simple systems, such as clusters of lattices with different uniform long-range interactions. We have generalized the formalism for more complicated interactions, but it would be useful to explore this further. In particular, one expects many related phenomena in other self-interacting theories. Non-Abelian gauge theories such as  $\phi^4$  theory may be interesting arenas of study. Numerical simulations might also be particularly useful to study some of these effects in more complicated systems.

It would also be useful to attempt to see these effects experimentally. The case of two clusters of lattices might be realized by making the clusters very small, so that the spacing between lattice sites is much smaller than the range of the spin-spin coupling.

## ACKNOWLEDGMENTS

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## APPENDIX A: THE DEFINITION OF LOCAL TEMPERATURE

In Sec. II B, we derived the distribution of the microlocal ensemble by looking at a system  $S$  and reservoir  $\mathcal{R}$  in the microcanonical ensemble. The local temperature was defined as

$$\beta_E \equiv \frac{\partial S_{\mathcal{R}}(E)}{\partial E}, \quad (\text{A1})$$

i.e., it was defined in terms of the entropy of the reservoir. In the noninteracting case, one tends to think of the temperature as

$$\beta'_E \equiv \frac{\partial S_S(E_S)}{\partial E_S}. \quad (\text{A2})$$

Our result of Sec. II F show that  $\beta_E \neq \beta'_E$ . The definitions are equivalent in the case where  $S$  is just a smaller part of a much larger system, i.e., when we formally divide a large system into  $\mathcal{R}$  and  $S$ . In this case, one can show

$$\beta_E = \beta'_E \quad (\text{A3})$$

because of symmetry.

To see this, we write the total energy of the total system as

$$m(E) = E + G(E), \quad (\text{A4})$$

where  $E = E_{\mathcal{R}} + E_{\mathcal{S}}$  as before. Then taking the partial derivative of Eq. (A4) with respect to  $E_{\mathcal{S}}$  and holding  $m$  fixed, we find

$$\begin{aligned} \left( \frac{\partial E_{\mathcal{R}}}{\partial E_{\mathcal{S}}} \right)_m &= -1 + \left( \frac{\partial G(E)}{\partial E_{\mathcal{S}}} \right)_m \\ &= -1 + \left( \frac{\partial G(E)}{\partial E_{\mathcal{R}}} \right)_m \\ &= -1. \end{aligned} \quad (\text{A5})$$

Combining this with Eq. (19), we obtain the desired result Eq. (A3).

Other special cases are when the reservoir has no long-range interactions or when the system has no long-range interactions. In the former case, one has

$$\beta_E = \beta'_E \left[ 1 - \left( \frac{\partial G(E_{\mathcal{S}})}{\partial E_{\mathcal{S}}} \right)_m \right], \quad (\text{A6})$$

and in the latter case

$$\beta_E = \beta'_E \left[ 1 - \left( \frac{\partial G(E_{\mathcal{R}}(E_{\mathcal{S}}))}{\partial E_{\mathcal{S}}} \right)_m \right]. \quad (\text{A7})$$

Both these results follow from Eq. (19). Note that in the former case, one also has that there is only one term in the sum in Eq. (15). This is because in this case,  $E_{\mathcal{R}}$  is uniquely determined from  $E_{\mathcal{R}} = m - E_{\mathcal{S}} - G(E_{\mathcal{S}})$ .

## APPENDIX B: A TOLMAN RELATION IN NEWTONIAN GRAVITY

We have seen in the case of the long-range lattice model that it has behavior reminiscent of a Schwarzschild geometry. This indicates that many of the thermodynamic properties one associates with general relativity may be present in Newtonian gravity. Indeed, we will now see that a Newtonian-type interaction does lead to the Tolman relation. Here, we will see that it arises from the long-range interactions and not necessarily from the curvature of space-time.

We imagine that we have two gravitating systems with mass  $M_1$  and  $M_2$ , and thermal energy  $E_1$  and  $E_2$  (which is the additional kinetic energy present in the molecules of each system), and we imagine that their volumes are fixed. The systems are assumed to be a distance  $d$  apart but in thermal contact (one might imagine that there is a conducting wire connecting the two systems). We will consider the following Newtonian-type interaction:

$$\begin{aligned} m &= M_1 + E_1 + M_2 + E_2 - G_1(M_1 + E_1)^2 - G_2(M_2 + E_2)^2 \\ &\quad - G_{12}(M_1 + E_1)(M_2 + E_2), \end{aligned} \quad (\text{B1})$$

where  $G$  are the coupling constants. Note that the model uses the fact that thermal energy also gravitates. Using Eq. (19) leads to a temperature ratio of

$$\frac{T_1}{T_2} = \frac{1 - G_{12}(M_2 + E_2) - 2G_1(M_1 + E_1)}{1 - G_{12}(M_1 + E_1) - 2G_2(M_2 + E_2)}. \quad (\text{B2})$$

Examples of systems which have such an interaction include two gravitating spheres separated by a distance  $d$ , as well as two concentric spherical shells at radii  $r_1$  and  $r_2$ . In the latter case, we have  $G_i = G/2r_i$  and  $G_{12} = G/r_2$ , where  $G$  is Newton's constant. To first order in the coupling constants, the temperature difference for two shells can then be written as

$$\left( \frac{T_1}{T_2} \right)_{shells} = 1 - G(M_1 + E_1) \left( \frac{1}{r_1} - \frac{1}{r_2} \right). \quad (\text{B3})$$

One can use a completely independent method to calculate the temperature ratio in full general relativity for this case, and one finds that the results are identical for weakly interacting fields where general relativity and Newtonian mechanics coincide. This, to a large extent, justifies the assumptions we made at the beginning of this paper.

Essentially, for gravity, the derived temperature difference coincides with what one expects from the Tolman relation, except in this case, there is a correction due to the fact that we are not considering a thermal system in a fixed gravitational background, but rather the thermal system is partly responsible for the gravitational interaction. For this reason, we see that the ratio does not only depend on the ratios of the redshifts  $1 - GM_i/d$ , but on  $1 - G(M_i + E_i)/d$ , i.e., the thermal energy  $E_i$  also contributes to the redshift factor.

One can also argue that the local temperature difference is indeed real for a freely falling observer, who essentially will be unaware of the additional gravitational interaction. Of course, this already invokes the equivalence principle. There are two other interesting points worth mentioning. One is that in order to get the temperature difference one needs to have a differences in charges (in this case, a difference between  $M_1$  and  $M_2$ ). It is this asymmetry which is partly responsible for the temperature difference. Additionally, one needs self-interactions, i.e., the thermal energy  $E_i$  needs to also gravitate. Thus, our model is not identical to Newtonian gravity, but includes the fact that all energy gravitates. This then seems to be the key ingredient which gives temperature differences.

## APPENDIX C: CONTINUUM LIMIT

In Sec. II F, we looked at the local temperature and global temperature of two systems in equilibrium. We then examined a simple example of two clusters interacting via two different uniform interactions. It is worthwhile to generalize this. For the case of a small number of regions, one can use the methods introduced earlier for just two regions. However, one can imagine a more complicated interaction like one of Eq. (3) where the interaction term is not a constant over any area, but instead changes from site to site. We can write instead  $J_{ij} = J(x_{ij})$  and then write all thermodynamics quantities as a function of the position  $x_{ij}$ . In fact, it will prove simplest to go to the case where we treat a system as a continuum—it is then easy to go back to the discrete case.

Let us now derive the relationships between the various local thermodynamical quantities. We essentially carry out a similar procedure as we did in Sec. II F. In other words, we extremize the total entropy at fixed total energy. We consider the entropy in terms of a density  $s(\rho(x))$ , where  $\rho$  can be thought of being related to a spin density. One can think of  $\rho$  as the local energy density in analogy to the quantity  $E$ , i.e., it is the continuum version of  $E$ . More explicitly, we can write  $\rho(x) = h(x)\sigma(x)$ , where  $\sigma(x)$  is the spin at each site  $x$  and  $h(x)$  is the energy gap at each site in the case where there is no interaction. We will however, leave it general, and simply write  $s(x)$  for simplicity (with the understanding that  $s$  is a function of the local energies), i.e.,

$$S = \int s(x) dx. \quad (C1)$$

We then extremize this by taking the variation and keeping  $m$  fixed. In order to do this, we append a constraint to the above expression, so that we instead extremize

$$L = S + \lambda \left( m - \int \dot{m} dx \right). \quad (C2)$$

Here, we are merely introducing the formalism. Indeed,  $\dot{m} \equiv dm/dx$  may be a complicated function, however, for the general lattice model of Eq. (3),  $\dot{m}$  is a functional of the spins at each site. It is in fact also functional of  $\rho$ . We can now vary  $L$  with respect to  $\delta\rho$ .

$$\begin{aligned} \delta L &= \int \left( \frac{\partial s}{\partial \rho} - \lambda \frac{\partial \dot{m}}{\partial \rho} \right) \delta \rho dx \\ &= \int \left( \beta(x) - \lambda \frac{\partial \dot{m}}{\partial \rho} \right) \delta \rho dx. \end{aligned} \quad (C3)$$

Since this must vanish for all  $\delta\rho$ , we have that

$$\beta(x) = \frac{\partial \rho}{\partial \dot{m}} \beta_o. \quad (C4)$$

Here, we have set the constant  $\lambda = \beta_o$ . One can see that in the case of no interactions, we have

$$m = \int \rho dx, \quad (C5)$$

and therefore

$$\dot{m} = \rho, \quad (C6)$$

so that we recover the standard result that the temperature is a constant. One can likewise obtain

$$\mu(\rho) \frac{\partial \rho}{\partial \dot{m}} = \mu_o. \quad (C7)$$

Here, the conserved energy can be defined as with  $E_o$  as  $\rho_o = (\partial \dot{m} / \partial \rho) \rho$  and as with Eq. (17), we have

$$\rho \beta = \beta_o \rho_o. \quad (C8)$$

Finally, we can find the variation in the remaining ‘‘intensive’’ quantity—the pressure  $p$  for systems which have particle flow. In order to let the system remain in mechanical equilibrium, the pressure will have to vary throughout a substance in order to keep the substance from flowing. This requirement gives

$$\frac{dp}{dx} = -F(x), \quad (C9)$$

where  $F(x)$  is the force due to the interaction. If the total  $m$  is simply some potential, then one would have  $F(x) = dm/dx$ .

#### APPENDIX D: GRAVITATING PERFECT FLUID

In this section, we will calculate the entropy of a spherically symmetric, self-gravitating perfect fluid. The field equations which govern the gravitating perfect fluid are well known [24]. Spherical symmetry implies that the metric takes the familiar form

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\Omega^2, \quad (D1)$$

where  $\Phi$  and  $\Lambda$  are functions of  $r$ . The stress-energy tensor of the perfect fluid is given in terms of the energy density  $\rho(r)$  and radial pressure  $p(r)$  by

$$T^{\mu\nu} = (\rho + p)u^\mu(r)u^\nu(r) + p(r)g^{\mu\nu}, \quad (D2)$$

where  $u^\mu(r)$  is the four-velocity of the fluid and  $g^{\mu\nu}$  is the metric. Einstein’s equation yield

$$e^{-2\Lambda} = 1 - 2m(r)/r, \quad (D3)$$

where

$$m(r) = \int_0^r 4\pi r'^2 \rho dr' \quad (D4)$$

and

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r(r - 2m)}. \quad (D5)$$

Outside the boundary of the fluid  $r = R$ , the functions  $\Lambda$  and  $\Phi$  reduce to

$$\begin{aligned} e^{-2\Lambda(R)} &= e^{2\Phi(R)} \\ &= 1 - 2M/R, \end{aligned} \quad (D6)$$

where  $M \equiv m(R)$ .

There are three conditions for equilibrium. If the system is in thermal equilibrium, it must obey the Tolman relation [10]

$$T(r) = T_o e^{-\Phi(r)}, \quad (D7)$$

where  $T_o$  is the temperature as measured at infinity. Likewise, the chemical potential at any two points can be related by the redshift to the value of the chemical potential on the boundary by

$$\mu(r) = \frac{\mu(R)e^{\Phi(R)}}{e^{\Phi(r)}}. \quad (\text{D8})$$

The condition for hydrostatic equilibrium (i.e., no radial infalling of any fluid element) can be found from local energy-momentum conservation

$$T_{;v}^{\mu\nu} = 0, \quad (\text{D9})$$

which implies

$$(\rho + p)\phi_{,r} = -p_{,r}. \quad (\text{D10})$$

For a perfect fluid of constant density, this leads to the well known Oppenheimer-Volkoff equation

$$\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}, \quad (\text{D11})$$

and simply balances the pressure gradient with the force due to gravity until equilibrium is reached.

We can now calculate the total entropy of the system. Since  $s$  is the local entropy as measured by observers in the rest frame of the fluid, we can integrate over the sphere to obtain the total entropy  $S$ . The appropriate volume element for a shell of thickness  $dr$  is  $dV = 4\pi r^2 e^\Lambda dr$  and so

$$\begin{aligned} S &= \int_0^R dV s(r) \\ &= \frac{4\pi}{T_o} \int_0^R r^2 e^{\Phi+\Lambda} [\rho + p - \mu_o n] dr, \end{aligned} \quad (\text{D12})$$

where we have used the Gibbs-Duhem and Tolman relations.

In general, we cannot solve this expression explicitly, however, for a perfect fluid with constant energy density  $\rho(r) = \rho_o$  and constant number density  $n(r) = n_o$ , the expressions for the metric and pressure are well known [19,25].

$$\begin{aligned} m(r) &= \begin{cases} (4\pi/3)\rho_o r^3 & r < R \\ M = (4\pi/3)\rho_o R^3 & r > R, \end{cases} \\ N(r) &= \begin{cases} (4\pi/3)n_o r^3 & r < R \\ N = (4\pi/3)n_o R^3 & r > R, \end{cases} \end{aligned}$$

$$e^\Phi = \frac{3}{2}k(R) - \frac{1}{2}k(r) \quad r < R, \quad (\text{D13})$$

$$p = \rho_o \frac{k(r) - k(R)}{3k(R) - k(r)} \quad r < R, \quad (\text{D14})$$

where  $k(r) \equiv \sqrt{1 - 2Mr^2/R^3}$ . It is worth observing that the pressure at  $r=0$  diverges as  $2M/R \rightarrow 8/9$  and as a result, this limits the size of our sphere of fluid.

We can now compute the entropy using Eq. (D12).

$$\begin{aligned} S &= \frac{4\pi}{T_o} \int_0^R r^2 dr \left[ \rho_o \frac{k(R)}{k(r)} - \mu_o n_o \frac{3k(R) - k(r)}{2k(r)} \right] \\ &= \frac{3k(R)R}{4T_o} \left( 1 - \frac{N\mu R}{M} \right) \left[ \sqrt{\frac{R}{2M}} \arcsin \sqrt{\frac{2M}{R}} - k(R) \right]. \end{aligned} \quad (\text{D15})$$

We see therefore that the entropy is no longer an extensive quantity and does not scale linearly with  $N$  and  $M$ , as it would for a system whose entropy scales like the volume of the system. However, if we expand our solution in terms of the gravitational coupling,  $M/R$ , then we find, to zeroth order in  $M/R$ ,

$$S \equiv [M - \mu(R)N]/T_o, \quad (\text{D16})$$

and we recover the extensive scaling of the entropy. To first order in  $M/R$

$$S \equiv [M - \mu(R)N] \left( 1 - \frac{2}{5} \frac{M}{R} \right) / T_p. \quad (\text{D17})$$

One would like to calculate the entropy for various equations of state. Unfortunately, one finds that for any realistic equation of state, the system is of infinite size. One can remedy this by having different equations of state at different radii, but this makes calculating the scaling behavior of the entropy completely meaningless, since it would depend more on how one changed the equations of state rather than on any properties of the states themselves. Another way of obtaining convergence, is to use the so-called ‘‘polytropic’’ equations of state such as

$$\rho = (\beta_o \mu_o + b) p^{a/(a+1)}. \quad (\text{D18})$$

However, they are not true equations of state and come from assuming that the matter is adiabatic as a function of  $r$ . They are therefore not suitable for states in thermal equilibrium.

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